

On the Geometry and Extremal Properties of the Edge-Degeneracy Model*

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Abstract

The edge-degeneracy model is an exponential random graph model that uses the graph degeneracy, a measure of the graph's connection density, and number of edges in a graph as its sufficient statistics. We show this model is relatively well-behaved by studying the statistical degeneracy of this model through the geometry of the associated polytope.

Keywords exponential random graph model, degeneracy, k -core, polytope

1 Introduction

Statistical network analysis is concerned with developing statistical tools for assessing, validating and modeling the properties of random graphs, or networks. The very first step of any statistical analysis is the formalization of a statistical model, a collection of probability distributions over the space of graphs (usually, on a fixed number of nodes n), which will serve as a reference model for any inferential tasks one may want to perform. Statistical models are in turn designed to be interpretable and, at the same time, to be capable of reproducing the network characteristics pertaining to the particular problem at hand. Exponential random graph models, or ERGMs, are arguably the most important class of models for networks with a long history. They are especially useful when one wants to construct models that resemble the observed network, but without the need to define an explicit network formation mechanism. In the interest of space, we single out classical references [2], [4], [9] and a recent review paper [8].

Central to the specification of an ERGM is the choice of sufficient statistic, a function on the space of graphs, usually vector-valued, that captures the particular properties of a network that are of scientific interest. Common examples of sufficient statistics are the number of edges, triangles, or k -stars, the degree sequence, etc; for an overview, see [8]. The choice of a sufficient statistic is not to be taken for granted: it depends on the application at hand and at the same time

it dictates the statistical and mathematical behavior of the ERGM. While there is not a general classification of 'good' and 'bad' network statistics, some lead to models that behave better asymptotically than others, so that computation and inference on large networks can be handled in a reliable way.

In an ERGM, the probability of observing any given graph depends on the graph only through the value of its sufficient statistic, and is therefore modulated by how much or how little the graph expresses those properties captured by the sufficient statistics. As there is virtually no restriction on the choice of the sufficient statistics, the class of ERGMs therefore possesses remarkable flexibility and expressive power, and offers, at least in principle, a broadly applicable and statistically sound means of validating any scientific theory on real-life networks. However, despite their simplicity, ERGMs are also difficult to analyze and are often thought to behave in pathological ways, e.g., give significant mass to extreme graph configurations. Such properties are often referred to as degeneracy; here we will refer to it as *statistical degeneracy* [10] (not to be confused with graph degeneracy below). Further, their asymptotic properties are largely unknown, though there has been some recent work in this direction; for example, [6] offer a variation approach, while in some cases it has been shown that their geometric properties can be exploited to reveal their extremal asymptotic behaviors [22], see also [18]. These types of results are interesting not only mathematically, but have statistical value: they provide a catalogue of extremal behaviors as a function of the model parameters and illustrate the extent to which statistical degeneracy may play a role in inference.

In this article we define and study the properties of the ERGM whose sufficient statistics vector consists of two quantities: the edge count, familiar to and often used in the ERGM family, and the graph degeneracy, novel to the statistics literature. (These quantities may be scaled appropriately, for purpose of asymptotic considerations; see Section 2.) As we will see, graph degeneracy arises from the graph's core structure, a property that is new to the ERGM framework [11], but is a natural connectivity statistic that gives a sense of how densely connected the most important actors in the network are. The core structure of a graph

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(see Definition 2.1) is of interest to social scientists and other researchers in a variety of applications, including the identification and ranking of influencers (or “spreaders”) in networks (see [14] and [1]), examining robustness to node failure, and for visualization techniques for large-scale networks [5]. The degeneracy of a graph is simply the statistic that records the largest core.

Cores are used as descriptive statistics in several network applications (see, e.g., [16]), but until recently, very little was known about statistical inference from this type of graph property: [11] shows that cores are unrelated to node degrees and that restricting graph degeneracy yields reasonable core-based ERGMs. Yet, there are currently no rigorous statistical models for networks in terms of their degeneracy. The results in this paper thus add a dimension to our understating of cores by exhibiting the behavior of the joint edge-degeneracy statistic within the context of the ERGM that captures it, and provide extremal results critical to estimation and inference for the edge-degeneracy model.

We define the edge-degeneracy ERGM in Section 2, investigate its geometric structure in Sections 3, 4, and 5, and summarize the relevance to statistical inference in Section 6.

2 The edge-degeneracy (ED) model

This section presents the necessary graph-theoretical tools, establishes notation, and introduces the ED model. Let \mathcal{G}_n denote the space of (labeled, undirected) simple graphs on n nodes, so $|\mathcal{G}_n| = 2^{\binom{n}{2}}$.

To define the family of probability distributions over \mathcal{G}_n comprising the ED model, we first define the degeneracy statistic.

DEFINITION 2.1. *Let $G = (V, E)$ be a simple, undirected graph. The k -core of G is the maximal subgraph of G with minimum degree at least k . Equivalently, the k -core of G is the subgraph obtained by iteratively deleting vertices of degree less than k . The graph degeneracy of G , denoted $\text{degen}(G)$, is the maximum value of k for which the k -core of G is non-empty.*

This idea is illustrated in Figure 1, which shows a graph G and its 2-core. In this case, $\text{degen}(G) = 4$.

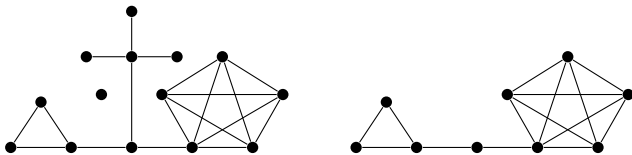


Figure 1: A small graph G (left) and its 2-core (right). The degeneracy of this graph is 4.

The *edge-degeneracy* ERGM is the statistical model on \mathcal{G}_n whose sufficient statistics are the rescaled graph degeneracy and the edge count of the observed graph. Concretely, for $G \in \mathcal{G}_n$ let

$$(2.1) \quad t(G) = \left(E(G)/\binom{n}{2}, \text{degen}(G)/(n-1) \right),$$

where $E(G)$ is the number of edges of G . The ED model on \mathcal{G}_n is the ERGM $\{P_{n,\theta}, \theta \in \mathbb{R}^2\}$, where

$$(2.2) \quad P_{n,\theta}(G) = \exp \{ \langle \theta, t(G) \rangle - \psi(\theta) \}$$

is the probability of observing the graph $G \in \mathcal{G}_n$ for the choice of model parameter $\theta \in \mathbb{R}^2$. The log-partition function $\psi: \mathbb{R}^2 \rightarrow \mathbb{R}$, given by $\psi(\theta) = \sum_{G \in \mathcal{G}_n} e^{\langle \theta, t(G) \rangle}$ serves as a normalizing constant, so that probabilities add up to 1 for each choice of θ (notice that $\psi(\theta) < \infty$ for all θ , as \mathcal{G}_n is finite).

Notice that different choices of $\theta = (\theta_1, \theta_2)$ will lead to rather different distributions. For example, for large and positive values of θ_1 and θ_2 the probability mass concentrates on dense graphs, while negative values of the parameters will favor sparse graphs. More interestingly, when one parameter is positive and the other is negative, the model will favor configurations in which the edge and degeneracy count will be balanced against each other. Our results in Section 5 will provide a catalogue of such behaviors in extremal cases and for large n .

The normalization of the degeneracy and the edge count in (2.1) and the presence of the coefficient $\binom{n}{2}$ in the ED probabilities (2.2) are to ensure a non-trivial limiting behavior as $n \rightarrow \infty$, since $E(G)$ and $\text{degen}(G)$ scale differently in n (see, e.g., [6] and [22]). This normalization is not strictly necessary for our theoretical results to hold. However, the ED model, like most ERGMs, is not consistent, thus making asymptotic considerations somewhat problematic.

LEMMA 2.1. *The edge-degeneracy model is an ERGM that is not consistent under sampling, as in [21].*

Proof. The range of graph degeneracy values when going from a graph with n vertices to one with $n+1$ vertices depends on the original graph; e.g. if there is a 2-star that is not a triangle in a graph with three vertices, the addition of another vertex can form a triangle and increase the graph degeneracy to 2, but if there was not a 2-star then there is no way to increase the graph degeneracy. Since the range is not constant, this ERGM is not consistent under sampling.

Thus, as the number of vertices n grows, it is important to note the following property of the ED model, not

uncommon in ERGMs: inference on the whole network cannot be done by applying the model to subnetworks.

In the next few sections we will study the geometry of the ED model as a means to derive some of its asymptotic properties. The use of polyhedral geometry in the statistical analysis of discrete exponential families is well established: see, e.g., [2], [4], [7], [20], [19].

3 Geometry of the ED model polytope

The edge-degeneracy ERGM (2.2) is a discrete exponential family, for which the geometric structure of the model carries important information about parameter estimation including existence of maximum likelihood estimate (MLE) - see above mentioned references. This geometric structure is captured by the *model polytope*.

The model polytope \mathcal{P}_n of the ED model on \mathcal{G}_n is the convex hull of the set of all possible edge-degeneracy pairs for graphs in \mathcal{P}_n . In symbols,

$$\mathcal{P}_n := \text{conv} \left\{ (E(G), \text{degen}(G)), G \in \mathcal{G}_n \right\} \subset \mathbb{R}^2.$$

Note the use of the unscaled version of the sufficient statistics in defining the model polytope. In this section, the scaling used in model definition (2.1) has little impact on shape of \mathcal{P}_n , thus - for simplicity of notation - we do not include it in the definition of \mathcal{P}_n . The scaling factors will be re-introduced, however, when we consider the normal fan and the asymptotics in Section 4.

In the following, we characterize the geometric properties of \mathcal{P}_n that are crucial to statistical inference. First, we arrive at a startling result, Proposition 3.2, that every integer point in the model polytope is a realizable statistic. Second, Proposition 3.4 implies that the observed network statistics will with high probability lie in the relative interior of the model polytope, which is an important property because estimation algorithms are guaranteed to behave well when off the boundary of the polytope. This also implies that the MLE for the edge-degeneracy ERGM exists for many large graphs. In other words, there are very few network observations that can lead to statistical degeneracy, that is, bad behavior of the model for which some ERGMs are famous. That behavior implies that a subset of the natural parameters is non-estimable, making complete inference impossible. Thus it being avoided by the edge-degeneracy ERGM is a desirable outcome. In summary, Propositions 3.4, 3.2, 3.5 and Theorem 3.1 completely characterize the geometry of \mathcal{P}_n and thus solve [17, Problem 4.3] for this particular ERGM. Remarkably, this problem—although critical for our understanding of reliability of inference for such models—has not been solved for most ERGMs except, for example, the beta model [20], which relied heavily on known graph-theoretic results.

Let us consider \mathcal{P}_n for some small values of n . The polytope \mathcal{P}_{10} is plotted in Figure 2.

The case $n = 3$. There are four non-isomorphic graphs on 3 vertices, and each gives rise to a distinct edge-degeneracy vector:

$$\begin{aligned} t \left(\begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \end{array} \right) &= (0, 0) & t \left(\begin{array}{ccc} & \bullet & \\ \bullet & & \bullet \end{array} \right) &= (1, 1) \\ t \left(\begin{array}{ccc} & \bullet & \\ \bullet & \bullet & \bullet \end{array} \right) &= (2, 1) & t \left(\begin{array}{ccc} & \bullet & \\ \bullet & \bullet & \bullet \end{array} \right) &= (3, 2) \end{aligned}$$

Hence $\mathcal{P}_3 = \text{conv} \{(0, 0), (1, 1), (2, 1), (3, 2)\}$. Note that in this case, each realizable edge-degeneracy vector lies on the boundary of the model polytope. We will see below that $n = 3$ is the unique value of n for which there are no realizable edge-degeneracy vectors contained in the relative interior of \mathcal{P}_n .

The case $n = 4$. On 4 vertices there are 11 non-isomorphic graphs but only 8 distinct edge-degeneracy vectors. Without listing the graphs, the edge-degeneracy vectors are:

$$(0, 0), (1, 1), (2, 1), (3, 1), (3, 2), (4, 2), (5, 2), (6, 3).$$

Here we pause to make the simple observation that $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ always holds. Indeed, every realizable edge-degeneracy vector for graphs on n vertices is also realizable for graphs on $n + 1$ vertices, since adding a single isolated vertex to a graph affects neither the number of edges nor the graph degeneracy.

The case $n = 5$. There are 34 non-isomorphic graphs on $n = 5$ vertices but only 15 realizable edge-degeneracy vectors. They are:

$$\begin{aligned} (0, 0), (1, 1), (2, 1), (3, 1), (3, 2), (4, 2), (5, 2), (6, 3), \\ (4, 1), (6, 2), (7, 2), (7, 3), (8, 3), (9, 3), (10, 4), \end{aligned}$$

where the pairs listed on the top row are contained in \mathcal{P}_4 and the pairs on the second row are contained in $\mathcal{P}_5 \setminus \mathcal{P}_4$. Here we make the observation that the proportion of realizable edge-degeneracy vectors lying on the interior of the \mathcal{P}_n seems to be increasing with n . This phenomenon is addressed in Proposition 3.4 below. Figure 2 depicts the integer points that define \mathcal{P}_{10} .

The case for general n . Many of the arguments below rely on the precise values of the coordinates of the extreme points of \mathcal{P}_n .

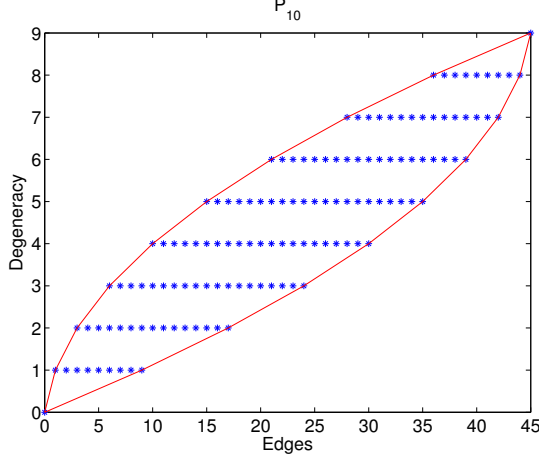


Figure 2: The integer points that define the model polytope \mathcal{P}_{10} .

PROPOSITION 3.1. Let $U_n(d)$ be the minimum number of edges over all graphs on n vertices with degeneracy d and let $L_n(d)$ be the maximum number of edges over all such graphs. Then,

$$U_n(d) = \binom{d+1}{2}$$

and

$$L_n(d) = \binom{d+1}{2} + (n-d-1) \cdot d.$$

Proof. First, observe that if $\text{degen}(G) = d$, then there are at least $d+1$ vertices of G in the d -core. Using this observation, it is not difficult to see that $U_n(d) = \binom{d+1}{2}$, since this is the minimum number of edges required to construct a graph with a non-empty d -core. Hence, the upper boundary of \mathcal{P}_n consists of the points $(\binom{d+1}{2}, d)$ for $0 \leq d \leq n-1$. For the value of $L_n(d)$, it is an immediate consequence of [11, Proposition 11] that $L_n(d) = \binom{d+1}{2} + (n-d-1) \cdot d$.

We use the notation $L_n(d)$ and $U_n(d)$ to signify that the extreme points $(L_n(d), d)$ and $(U_n(d), d)$ lie on the lower and upper boundaries of \mathcal{P}_n , respectively.

It is well known in the theory of discrete exponential families that the MLE exists if and only if the average sufficient statistic of the sample lies in the relative interior of the model polytope. This leads us to investigate which pairs of integer points correspond to realizable edge-degeneracy vectors.

PROPOSITION 3.2. Every integer point contained in \mathcal{P}_n is a realizable edge-degeneracy vector.

Proof. Suppose that G is a graph on n vertices with $\text{degen}(G) = d \leq n-1$. Our strategy will be to show that for all e such that $U_n(d) \leq e \leq L_n(d)$, there exists a graph G on n vertices such that $\text{degen}(G) = d$ and $E(G) = e$.

It is clear that there is exactly one graph (up to isomorphism) corresponding to the edge-degeneracy vector $(\binom{d+1}{2}, d)$; it is the graph

$$(3.3) \quad K_{d+1} \cup \underbrace{K_1 \cup \dots \cup K_1}_{n-d-1 \text{ times}},$$

i.e., the complete graph on $d+1$ vertices along with $n-d-1$ isolated vertices.

Thus, for each e such that $U_n(d) = \binom{d+1}{2} < e < \binom{d+1}{2} + (n-d-1) \cdot d = L_n(d)$, we must show how to construct a graph G on n vertices with graph degeneracy d and e edges. Call the resulting graph $G_{n,d,e}$. To construct $G_{n,d,e}$, start with the graph in 3.3, which has degeneracy d . Label the isolated vertices v_1, \dots, v_{n-d-1} . For each j such that $1 \leq j \leq e - \binom{d+1}{2}$, add the edge e_j by making the vertex v_i such that $i \equiv j \pmod{n-d-1}$ adjacent to an arbitrary vertex of K_{d+1} . This process results in a graph with exactly $e = \binom{d+1}{2} + j$ edges, and since the vertices v_1, \dots, v_{n-d-1} still have degree at most d , our construction guarantees that we have not increased the graph degeneracy. Hence, we have constructed $G_{n,d,e}$.

The preceding proof also shows the following:

PROPOSITION 3.3. \mathcal{P}_n contains exactly

$$(3.4) \quad \sum_{d=0}^{n-1} [(n-d-1) \cdot d + 1]$$

integer points. This is the number of realizable edge-degeneracy vectors for every n .

The following property is useful throughout:

LEMMA 3.1. \mathcal{P}_n is rotationally symmetric.

Proof. For $n \geq 3$ and $d \in \{1, \dots, n-1\}$,

$$L_n(d) - L_n(d-1) = U_n(n-d) - U_n(n-d-1).$$

Note that the center of rotation is the point $((n-1)n/4, (n-1)/2)$. The rotation is 180 degrees around that point.

As mentioned above, the following nice property of the ED model polytope-in conjunction with the partial characterization of the boundary graphs below- suggests that the MLE for the ED model exists for most large graphs.

PROPOSITION 3.4. *Let p_n denote the proportion of realizable edge-degeneracy vectors that lie on the relative interior of \mathcal{P}_n . Then,*

$$\lim_{n \rightarrow \infty} p_n = 1.$$

Proof. This result follows from analyzing the formula in 3.4 and uses the following lemma:

LEMMA 3.2. *There are $2n-2$ realizable lattice points on the boundary of \mathcal{P}_n , and each is a vertex of the polytope.*

Proof. Since $\mathcal{P}_n \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$, and $(0,0) \in \mathcal{P}_n$ for all n , we know that $(0,0)$ must be a vertex of \mathcal{P}_n . By the rotational symmetry of \mathcal{P}_n , $(n-1, (n-1)n/2)$ must be a vertex, too. It is clear that the points of the form $(U_n(d), d)$ and $(L_n(d), d)$ for $d \in \{1, \dots, n-2\}$ are the only other points on the boundary; we will show that each of these points is in fact a vertex. For this it suffices to observe that $U_n(d)$ is strictly concave and $L_n(d)$ is strictly convex as a function of d . Hence, no interval $[U_n(d), L_n(d)]$ is contained in the convex hull of any collection of intervals of the same form. Thus, for each $d \in \{1, \dots, n-2\}$ the points $(U_n(d), d)$ and $(L_n(d), d)$ are vertices of \mathcal{P}_n .

To prove Proposition 3.4, we then compute:

$$p_n = \frac{\sum_{d=0}^{n-1} [(n-d-1) \cdot d + 1] - (2n-2)}{\sum_{d=0}^{n-1} [(n-d-1) \cdot d + 1]} \rightarrow 1.$$

3.1 Extremal graphs of \mathcal{P}_n . Now we turn our attention to the problem of identifying the graphs corresponding to extreme points of \mathcal{P}_n . Clearly, the boundary point $(0,0)$ is uniquely attained by the empty graph \bar{K}_n and the boundary point $(\binom{n}{2}, n-1)$ is uniquely attained by the complete graph K_n . The proof of Proposition 3.2 shows that the unique graph corresponding to the upper boundary point $(U_n(d), d)$ is a complete graph on $d+1$ vertices union $n-d-1$ isolated vertices. The lower boundary graphs are more complicated, but the graphs corresponding to two of them are classified in the following proposition.

PROPOSITION 3.5. *The graphs corresponding to the lower boundary point $(L_n(1), 1)$ of \mathcal{P}_n are exactly the trees on n vertices. The unique (up to isomorphism) graph corresponding to the lower boundary point $(L_n(n-2), n-2)$ is the complete graph on n vertices minus an edge.*

Proof. First we consider lower boundary graphs with edge-degeneracy vector $(L_n(1), 1)$. A graph with degeneracy 1 must be acyclic, since otherwise it would have degeneracy at least 2. Hence, such a graph must

be a forest. However, if the forest is not connected, one could add an edge without increasing the degeneracy, and thus it must be a tree. For the second statement, the complete graph minus one edge has the most edges among all non-complete graphs. Any other graph has either larger degeneracy or fewer edges.

The graphs corresponding to extreme points $(L_n(d), d)$ for $2 \leq d \leq n-3$ are called *maximally d -degenerate* graphs and were studied extensively in [3]. Such graphs have many interesting properties, but are quite difficult to fully classify or enumerate.

In the following theorem, we show that the lower boundary of \mathcal{P}_n is like a mirrored version of the upper boundary. This partially characterizes the remaining maximally d -degenerate graphs.

THEOREM 3.1. *Let $G(U_n(d))$ to be the unique (up to isomorphism) graph on n nodes with degeneracy d that has the minimum number of edges, given by $U_n(d)$. Similarly, let $G(L_n(d)) \subset \mathcal{G}_n$ be the set of graphs on n nodes with degeneracy d that have the maximum number of edges, given by $L_n(d)$. Then, for all $d \in \{0, 1, \dots, n-1\}$,*

$$\overline{G(U_n(d))} \in G(L_n(n-d-1)),$$

where $\overline{G(U_n(d))}$ denotes the graph complement of $G(U_n(d))$.

Proof. As we know, $G(U_n(d)) = K_{d+1} \cup \bar{K}_{n-d-1}$. Taking the complement,

$$(3.5) \quad \overline{G(U_n(d))} = \bar{K}_{d+1} + K_{n-d-1},$$

where $+$ denotes the *graph join* operation. We only need to show that this has $L_n(n-d-1)$ edges and graph degeneracy $n-d-1$.

Since $G(U_n(d))$ has $\binom{d+1}{2}$ edges, $\overline{G(U_n(d))}$ must have $\binom{n}{2} - \binom{d+1}{2} = L_n(n-d-1)$ edges. As for the graph degeneracy, $K_1 + K_{n-d-1} = K_{n-d}$ is a subgraph of $\overline{G(U_n(d))}$. Therefore, $\text{degen}(\overline{G(U_n(d))}) \geq n-d-1$. However, $\text{degen}(\overline{G(U_n(d))}) < n-d$ because there are $d+1$ vertices of degree $n-d-1$, and a non-empty $(n-d)$ -core would require at least $n-d+1$ vertices of degree at least $n-d$. Thus, $\text{degen}(\overline{G(U_n(d))}) = n-d-1$, as desired.

4 Asymptotics of the ED model polytope and its normal fan

Since we will let $n \rightarrow \infty$ in this section, it will be necessary to rescale the polytope \mathcal{P}_n so that it is contained in $[0, 1]^2$, for each n . Thus, we divide the graph degeneracy parameter by $n-1$ and the edge parameter by $\binom{n}{2}$, as we have already done in (2.1).

While this rescaling has little impact on shape of \mathcal{P}_n discussed in Section 3, it does affect its normal fan, a key geometric object that plays a crucial role in our subsequent analysis. We describe the normal fan of the normalized polytope next.

PROPOSITION 4.1. *All of the perpendicular directions to the faces of \mathcal{P}_n are:*

$$\{\pm(1, -m) : m \in \{1, 2, \dots, n-1\}\}.$$

So, after normalizing, we get that the directions are

$$\{\pm(1, -\frac{2}{\alpha n}) : \alpha \in \{1/(n-1), 2/(n-1), \dots, 1\}\}.$$

Proof. The slopes of the line segments defining each face of \mathcal{P}_n are $1/\Delta_U(d) = 1/d$ in the unnormalized parametrization. To get the slopes of the normalized polytope, just multiply each slope by $\binom{n}{2}/(n-1) = n/2$.

Our next goal is to describe the limiting shape of the normalized model polytope and its normal fan as $n \rightarrow \infty$. We first collect some simple facts about the limiting behavior of the normalized graph degeneracy and edge count.

PROPOSITION 4.2. *If $\alpha \in [0, 1]$ such that $\alpha(n-1) \in \mathbb{N}$ (so that α parameterizes the normalized graph degeneracy), then*

$$\lim_{n \rightarrow \infty} \frac{U_n(\alpha(n-1))}{\binom{n}{2}} = \alpha^2.$$

Furthermore, due to the rotational symmetry of \mathcal{P}_n ,

$$\lim_{n \rightarrow \infty} \frac{L_n(\alpha(n-1))}{\binom{n}{2}} = 1 - (1 - \alpha)^2.$$

Proof. By definition, $U_n(\alpha(n-1)) = \frac{\alpha^2}{2}n^2 + o(n)$. Hence,

$$\frac{U_n(\alpha(n-1))}{\binom{n}{2}} = \frac{\frac{\alpha^2}{2}n^2 + o(n)}{\frac{n(n-1)}{2}} \rightarrow \alpha^2.$$

We can now proceed to describe the set limit corresponding to the sequence $\{\mathcal{P}_n\}_n$ of model polytopes. Here, the notion of set limit is the same as in [22, Lemma 4.1]. Let

$$\mathcal{P} = \text{cl} \{t \in \mathbb{R}^2 : t = t(G), G \in \mathcal{G}_n, n = 1, 2, \dots\}$$

be the closure of the set of all possible realizable statistics (2.1) from the model. Using Propositions 4.1 and 4.2 we can characterize \mathcal{P} as follows.

LEMMA 4.1. 1. $\mathcal{P}_n \subset \mathcal{P}$ for all n and $\lim_n \mathcal{P}_n = \mathcal{P}$.

2. Let L and U be functions from $[0, 1]$ into $[0, 1]$ given by

$$L(x) = 1 - \sqrt{1-x} \quad \text{and} \quad U(x) = \sqrt{x}.$$

Then,

$$\mathcal{P} = \{(x, y) \in [0, 1]^2 : L(x) \leq y \leq U(x)\}.$$

Proof. If $\alpha \in [0, 1]$ is such that $\alpha(n-1) \in \mathbb{N}$, then

$$\frac{U_n(\alpha(n-1))}{\binom{n}{2}} = \alpha^2 + \frac{\alpha - \alpha^2}{n} \geq \alpha^2$$

for any finite value of n . Similarly,

$$\frac{L_n(\alpha(n-1))}{\binom{n}{2}} = 2\alpha - \alpha^2 + \frac{\alpha^2 - \alpha}{n} \leq 1 - (1 - \alpha)^2,$$

for any finite value of n . Hence, for any n ,

$$\alpha^2 \leq \frac{U_n(\alpha(n-1))}{\binom{n}{2}} \leq \frac{L_n(\alpha(n-1))}{\binom{n}{2}} \leq 1 - (1 - \alpha)^2.$$

Together with Propositions 4.1 and 4.2, this completes the proof.

The convex set \mathcal{P} is depicted at the top of Figure 4.

In order to study the asymptotics of extremal properties of the ED model, the final step is to describe all the normals to \mathcal{P} . As we will see in the next section, these normals will correspond to different extremal behaviors of the model. Towards this end, we define the following (closed, pointed) cones

$$\begin{aligned} C_\emptyset &= \text{cone} \{(1, -2), (-1, 0)\}, \\ C_{\text{complete}} &= \text{cone} \{(1, 0), (-1, 2)\}, \\ C_U &= \text{cone} \{(-1, 0), (-1, 2)\}, \\ C_L &= \text{cone} \{(1, 0), (1, -2)\}, \end{aligned}$$

where, for $A \subset \mathbb{R}^2$, $\text{cone}(A)$ denote the set of all conic (non-negative) combinations of the elements in A . It is clear that C_\emptyset and C_{complete} are the normal fan to the points $(0, 0)$ and $(1, 1)$ of \mathcal{P} . As for the other two cones, it is not hard to see that the set of all normal rays to the edges of the upper, resp. lower, boundary of \mathcal{P}_n for all n are dense in C_U , resp. C_L . As we will show in the next section, the regions C_\emptyset and C_{complete} indicate directions of statistical degeneracy (for large n) towards the empty and complete graphs, respectively. On the other hand, C_U and C_L contain directions of non-trivial convergence to extremal configurations of maximal and minimal graph degeneracy. See Figure 3 and the middle and lower part of Figure 4.

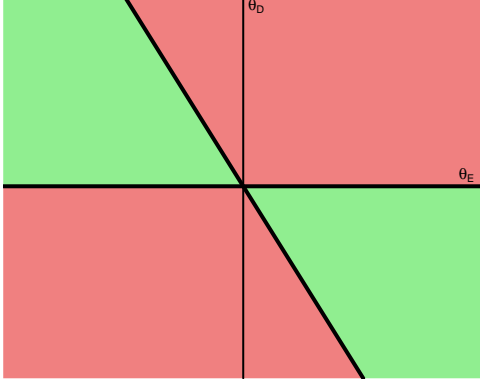


Figure 3: The green regions indicate directions of nontrivial convergence. The bottom-left and top-right red regions indicate directions towards the empty graph \bar{K}_n and complete graph K_n , respectively.

5 Asymptotical Extremal Properties of the ED Model

In this section we will describe the behavior of distributions from the ED model of the form $P_{n,\beta+rd}$, where d is a non-zero point in \mathbb{R}^2 and r a positive number. In particular, we will consider the case in which d and β are fixed, but n and r are large (especially r). We will show that there are four possible types of extremal behavior of the model, depending on d , the “direction” along which the distribution becomes extremal (for fixed n and as r grows unbounded). This dependence can be loosely expressed as follows: each d will identify one and only one value $\alpha(d)$ of the normalized edge-degeneracy pairs such that, for all n and r large enough, $P_{n,\beta+rd}$ will concentrate only on graphs whose normalized edge-degeneracy value is arbitrarily close to $\alpha(d)$.

In order to state this result precisely, we will first need to make some elementary, yet crucial, geometric observations. Any $d \in \mathbb{R}^2$ defines a normal direction to one point on the boundary of \mathcal{P} . Therefore, each $d \in \mathbb{R}^2$ identifies one point on the boundary of \mathcal{P} , which we will denote $\alpha(d)$. Specifically, any $d \in C_\emptyset$ is in the normal cone to the point $(0,0) \in \mathcal{P}$, so that $\alpha(d) = (0,0)$ (the normalized edge-degeneracy of the empty graph) for all $d \in C_\emptyset$ (and those points only). Similarly, any $d \in C_{\text{complete}}$ is in the normal cone to the point $(1,1) \in \mathcal{P}$, and therefore $\alpha(d) = (1,1)$ (the normalized edge-degeneracy of K_n) for all $d \in C_{\text{complete}}$ (and those points only). On the other hand, if $d \in \text{int}(C_L)$, then d is normal to one point on the upper boundary of \mathcal{P} . Assuming without loss of generality that $d = (1, a)$, $\alpha(d)$ is the point (x, y) along the curve $\{L(x), x \in [0, 1]\}$ such that $L'(x) = -\frac{1}{a}$. Notice that, unlike the previous cases, if d and d' are distinct points in $\text{int}(C_L)$ that are not

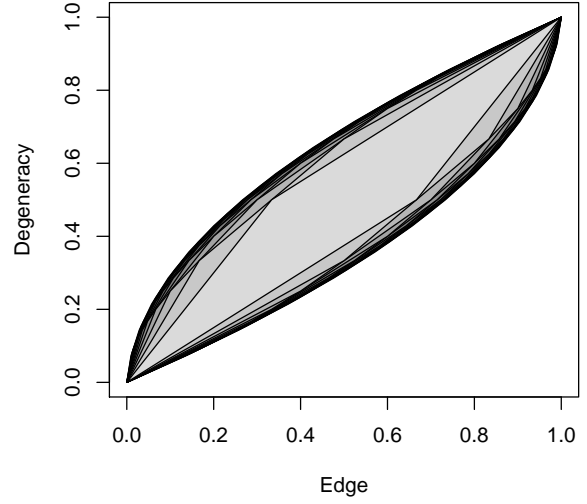


Figure 4: (Top) the sequence of normalized polytopes $\{\mathcal{P}_n\}_n$ converges outwards, starting from \mathcal{P}_3 in the center. (Middle) and (bottom) are the representative infinite graphs along points on the upper and lower boundaries of \mathcal{P} , respectively, depicted as graphons [15] for convenience.

collinear, $\alpha(d) \neq \alpha(d')$. Analogous considerations hold for the points $d \in \text{int}(C_U)$: non-collinear points map to different points along the curve $\{U(x), x \in [0, 1]\}$.

With these considerations in mind, we now present our main result about the asymptotics of extremal properties of the ED model.

THEOREM 5.1. *Let $d \neq 0$ and consider the following cases.*

- $d \in \text{int}(C_\emptyset)$.
Then, for any $\beta \in \mathbb{R}^2$ and arbitrarily small $\epsilon \in (0, 1)$ there exists a $n(\epsilon)$ such that for all $n \geq n(\epsilon)$ there exists a $r = r(\epsilon, n)$ such that, for all $r \geq$

$r(\epsilon, n)$ the empty graph has probability at least $1 - \epsilon$ under $P_{n, \beta+rd}$.

- $d \in \text{int}(C_{\text{complete}})$.
Then, for any $\beta \in \mathbb{R}^2$ and arbitrarily small $\epsilon \in (0, 1)$ there exists a $n(\epsilon)$ such that for all $n \geq n(\epsilon)$ there exists a $r = r(\epsilon, n)$ such that, for all $r \geq r(\epsilon, n)$ the complete graph has probability at least $1 - \epsilon$ under $P_{n, \beta+rd}$.
- $d \in \text{int}(C_L)$.
Then, for any $\beta \in \mathbb{R}^2$ and arbitrarily small $\epsilon, \eta \in (0, 1)$ there exists a $n(\epsilon)$ such that for all $n \geq n(\epsilon, \eta)$ there exists a $r = r(\epsilon, n)$ such that, for all $r \geq r(\epsilon, \eta, n)$ the set of graphs in \mathcal{G}_n whose normalized edge-degeneracy is within η of $\alpha(d)$ has probability at least $1 - \epsilon$ under $P_{n, \beta+rd}$.
- $d \in \text{int}(C_U)$.
Then, for any $\beta \in \mathbb{R}^2$ and arbitrarily small $\epsilon, \eta \in (0, 1)$ there exists a $n(\epsilon)$ such that for all $n \geq n(\epsilon, \eta)$ there exists a $r = r(\epsilon, n)$ such that, for all $r \geq r(\epsilon, \eta, n)$ the set of graphs in \mathcal{G}_n whose normalized edge-degeneracy is within η of $\alpha(d)$ has probability at least $1 - \epsilon$ under $P_{n, \beta+rd}$.

Remarks. We point out that the directions along the boundaries of C_\emptyset and C_{complete} are not part of our results. Our analysis can also accommodate those cases, but in the interest of space, we omit the results. More importantly, the value of β does not play a role in the limiting behavior we describe. We further remark that it is possible to formulate a version of Theorem 5.1 for each finite n , so that only r varies. In that case, by Proposition 4.1, for each n there will only be $2(n-1)$ possible extremal configurations, aside from the empty and fully connected graphs. We have chosen instead to let n vary, so that we could capture all possible cases.

Proof. We only sketch the proof for the case $d \in \text{int}(C_L)$, which follows easily from the arguments in [22], in particular Propositions 7.2, 7.3 and Corollary 7.4. The proofs of the other cases are analogous. First, we observe that the assumption (A1)-(A4) from [19] hold for the ED model. Next, let n be large enough such that d is not in the normal cone corresponding to the points $(0, 0)$ and $(1, 1)$ of \mathcal{P}_n . Then, for each such n , d either defines a direction corresponding to the normal of an edge, say e_n , of the upper boundary of \mathcal{P}_n or d is in the interior of the normal cone to a vertex, say v_n , of \mathcal{P}_n . Since $\mathcal{P}_n \rightarrow \mathcal{P}$, n can be chosen large enough so that either the vertices of e_n or v_n (depending on which one of the two cases we are facing) are within η of $\alpha(d)$.

Let us first consider the case when d is normal to the edge v_n of \mathcal{P}_n . Since every edge of \mathcal{P}_n contains

only two realizable pairs of normalized edge count and graph degeneracy, namely its endpoints, using the result in [22], one can choose $r = r(n, \epsilon, \eta)$ large enough so that at least $1 - \epsilon$ of the mass probability of $P_{n, \beta+rd}$ concentrates on the graphs in \mathcal{G}_n whose normalized edge-degeneracy vector is either one of the two vertices in e_n . The claim follows from the fact that these vertices are within η of $\alpha(q)$. For the other case in which d is in the interior of the normal cone to the vertex v_n , again the results in [22] yield that one can choose $r = r(n, \epsilon, \eta)$ large enough so that at least $1 - \epsilon$ of the mass probability of $P_{n, \beta+rd}$ concentrates on graphs in \mathcal{G}_k whose normalized edge-degeneracy vector is v_n . Since v_n is within η of $\alpha(q)$ we are done.

The interpretation of Theorem 5.1 is as follows. If d is a non-zero direction in C_\emptyset and C_{complete} , then $P_{n, \beta+rd}$ will exhibit statistical degeneracy regardless of β and for large enough r , in the sense that it will concentrate on the empty and fully connected graphs, respectively. As shown in Figure 3, C_\emptyset and C_{complete} are fairly large regions, so that one may in fact expect statistical degeneracy to occur prominently when the model parameters have the same sign. The extremal directions in C_U and C_L yields instead non-trivial behavior for large n and r . In this case, $P_{n, \beta+rd}$ will concentrate on graph configurations that are extremal in sense of exhibiting nearly maximal or minimal graph degeneracy given the number of edges.

Taken together, these results suggest that care is needed when fitting the ED model, as statistical degeneracy appears to be likely.

6 Discussion

The goal of this paper is to introduce a new ERGM and demonstrate its statistical properties and asymptotic behavior captured by its geometry. The ED model is based on two graph statistics that are not commonly used jointly and capture complementary information about the network: the number of edges and the graph degeneracy. The latter is extracted from important information about the network's connectivity structure called cores and is often used as a descriptive statistic.

The exponential family framework provides a beautiful connection between the model geometry and its statistical behavior. To that end, we completely characterized the model polytope in Section 3 for finite graphs and Section 4 for the limiting case as $n \rightarrow \infty$. The most obvious implication of the structure of the ED model polytope is that the MLE exists for a significant proportion of large graphs. Another is that it simplifies greatly the problem of projecting noisy data onto the polytope and finding the nearest realizable point, as one need

only worry about the projection. Such projections play a critical role in data privacy problems, as they are used in computing a private estimator of the released data with good statistical properties; see [12, 13]. Finally, the structure of the polytope and its normal fan reveal various extremal behaviors of the model, discussed in Section 5.

Note that the two statistics in the ED model summarize very different properties of the observed graph, giving this seemingly simple model some expressive power and flexibility. In graph-theoretic terms, the degeneracy summarizes the core structure of the graph, within which there can be few or many edges (see [11] for details); combining it with the number of edges produces Erdős-Renyi as a submodel.

As discussed in Section 2, different choices of the parameter vector, that is, values of the edge-degeneracy pair, lead to rather different distributions, from sparse to dense graphs as both parameters are negative or positive, respectively, as well as graphs where edge count and degeneracy are balanced for mixed-sign parameter vectors. Our results in Section 5 provide a catalogue of such behaviors in extremal cases and for large n . The asymptotic properties we derive offer interesting insights on the extremal asymptotic behavior of the ED model. However, the asymptotic properties of non-extremal cases, that is, those of distributions of the form $P_{n,\beta}$ for fixed β and diverging n , remain completely unknown. While this is an exceedingly common issue with ERGMs, whose asymptotics are extremely difficult to describe, it would nonetheless be desirable to gain a better understanding of the ED model when the network is large. In this regard, the variation approach put forward by [6], which provides a way to resolve the asymptotics of ERGMs in general, may be an interesting direction to pursue in future work.

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